SERIES WORKSHEET 1 SOLUTION SKETCHES

Note: These are not model solutions, but only sketches/hints towards solutions.

Problem 1. Decide whether the series converge absolutely, conditionally, or if they diverge.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!}$, (2) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{50}50^n}{n!}$, (3) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(-3)^n}$, (4) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+3)4^{2n-1}}$,

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$, (6) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$, (7) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{5}{3}}}$, (8) $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$,

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n!}}$, (10) $\displaystyle\sum_{n=1}^{\infty} e^{-\sqrt{n}}$, (11) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2}{e^n}-\frac{n^2}{1+n^3}\right)$, (12) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\ln n}$,

(13) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\sin\frac{1}{n}}}$, (14) $\displaystyle\sum_{n=1}^{\infty} \sin(e^{-n})$, (15) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{\ln(\ln n)}}}$, (16) $\displaystyle\sum_{n=1}^{\infty} e^{-(\ln n)^2}$,

Solution.

(1) Converges absolutely. Apply ratio test.

(2) Converges absolutely. Apply ratio test.

(3) Converges absolutely. Apply ratio or root test.

(4) Converges absolutely. Apply ratio or root test.

(5) Diverges. Terms don't go to 0, alternatively apply ratio or root test.

(6) Converges absolutely. Apply ratio test.

(7) Converges absolutely. Taking absolute values of the summands gives p-series with $p = \frac{5}{3} > 1$.

(8) Converges absolutely. Apply root test.

(9) Diverges. Compare $\sqrt[n]{n!} \leq \sqrt[n]{n^n} = n$.

(10) Converges. Apply integral test or compare with $\frac{1}{n^p}$ with any $p > 1$.

(11) Diverges. First term gives convergent series, second term gives divergent series, then the whole series diverges.

(12) Converges conditionally. Alternating test. To see the series does not converge absolutely use $\ln n \leq n$.

(13) Diverges. Do limit comparison test with $\frac{1}{n}$.

(14) Converges absolutely. Compare with e^{-n}.

(15) Converges absolutely. Do limit comparison test with $\frac{1}{n\ln^2(n)}$. (This one was tricky!)

(16) Converges absolutely. Do limit comparison test with $\frac{1}{n^p}$ for any $p > 1$.

Problem 2. Let $(a_n)_n, (b_n)_n$ be sequences of real numbers. Decide with justification (proof or counterexample) whether the statement is true:

(a) ? If $\sum_{n=1}^{\infty} |a_n|$ converges and $(b_n)_n$ is bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges ?

(b) ? If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ both diverge, then so does $\sum_{n=1}^{\infty} a_n + b_n$?

(c) ? If $\sum_{n=1}^{\infty} a_n$ converges, then the sequence $(na_n)_n$ is bounded ?

(d) ? If $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} a_n^2$?

Solution.

(a) True. If $|b_n| \leq C$ for all n, then $|a_n b_n| \leq C\,|a_n|$, so $\sum_{n=1}^{\infty} |a_n b_n|$ converges by comparison, hence $\sum_{n=1}^{\infty} a_n b_n$ converges since it converges absolutely.

(b) False. Take e.g. $a_n = 1, b_n = -1$ for all n.

(c) False. Take e.g. $a_n = \frac{(-1)^n}{\sqrt{n}}$. (There are also examples with all $a_n \geq 0$. Find one!)

(d) True. Since $\sum_{n=1}^{\infty} a_n$ converges, eventually $a_n \leq 1$. Then for those n we have $0 \leq a_n^2 \leq a_n$, so $\sum_{n=1}^{\infty} a_n^2$ converges by comparison.

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